Scale Free Controllability of Large-Scale Networks: an Output Controllability Approach

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***Abstract*—In this paper we consider the problem of controllability and energy consumption for large scale networks. Instead of controlling separately all the nodes of the network we control an output which is defined as some measurement (for instance the average) of the nodes which are not directly controlled. We thus exploit the concept of Output Controllability and the Output Controllability Gramian to analyze the properties of the system. In this context, we show that it is possible to obtain a reduced-order model which makes the Gramian compution and control design much easier. Simulations show that the reduced model is consistent with the original one and for low ratios of controlled nodes, more robust and performing with respect to the original.**

I. Introduction

In recent years, the interest in the topic of networks control have been growing. This interest is motivated by the vast number of applications which nowadays can be framed into the context of networks. Robot swarms ([2]), power networks ([3]) and social networks ([4]), are just few examples which pushed researchers in studying control problems for networks. Many results about consensus and synchronization can nowadays be found both in the context of linear ([5], [6], [7], [12]) and nonlinear systems ([8], [9], [10], [11], [13]). In recent years, due to the increasing complexity of networks applications, the focus has shifted towards the framework of large scale networks. The analysis of large scale networks has been widely covered in the literature and notable contributions can be found in [14], [15], [16]. In particular, in [17], [18], it has been observed that many real world large networks such as the world wide web present a *scaling effect* in their growth, namely the connectivity degree follows a power law distribution.

The problem of controlling large scale networks is with many extents still an open problems. A fundamental aspect that has to be considered is the energy necessary to control a system and with this respect the analysis of large networks have been addressed by several authors ([19], [20], [21]). It has been shown that several aspects such as network centrality [22], [23], exact controllability [24], [25] and number of control nodes [26] plays a fundamental role in assessing controllability. It is however important to distinguish between the classic controllability concept and practical controllability. Classic controllability is defined as a property of the state and input maps while practical controllability is related to the energy necessary to control the system. With this respect, in [27] the authors have shown that an anlysis of the practical controllability of a large network is really hard to perform.

This work is carried out within the context of the ERC, Scale-FreeBack (see [1]) which aims to develop holistic scale-free modeling, estimation and control methods for complex network systems. Scale-free control refers to the ability of a control algorithm to easily scale to any network dimension. Several large-scale systems are described by *homogeneous* networks in which nodes have practically the same degree and are inherently difficult to model and control. These networks can however be abstracted/aggregated/reduced to a Scale-Free graph characterized by heavy-tailed node/link distributions obeying a power law, for which the underlying network structure is dominated by relatively small numbers of nodes (hubs) having a large number of connections. It is expected that controlling scale-free networks by acting only on a limited number of lumped nodes, the control design for large-scale systems will be tractable. This idea boils dawn to control some few outputs (for instance representing the average value of a large group of states), rather than trying to control the network full state. In this context, we can adapt the conventional controllability to the concept of output controllability and consequently the Controllability Gramian to the Output Controllability Gramian. The latter allows to evaluate the controllability and the energy necessary to steer the output to a desired value. By its definition however, the Output Controllability Gramian still requires the computation of the Controllability Gramian and thus presents the problem mentioned previously.

To overcome this issue, in this paper we propose a output-based *model reduction* to evaluate the controllability of a large scale network. In particular, we define a reduced model which aggregates the dynamics which contributes to the output. The reduced-model simplifies the computation of the Controllability Gramian and the minimum energy control inputs to steer the ouput of the system to a desired value. Simulation results shows that this reduced system is more robust to noise and ill-conditionement of the Gramian and, for small ratios of controlled nodes, control design based on the reduced model performs better than the design based on the original system. Furthermore, thanks to the simplicity

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of the reduce system Gramian, a relationship between the control energy, the networks control-degree and *inertia* can be displayed.

II. Problem Formulation

Consider a system

$$\dot{x} = Lx + Bu \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $L \in \mathbb{R}^{n\times n}$ a stable matrix and $B \in \mathbb{R}^{n\times m}$. As far as controllability is concerned, we recall that the Controllability Gramian is defined as

$$\mathcal{W}_t = \int_{\tau=0}^{t} e^{L(t-\tau)}BB^T e^{L^T(t-\tau)}\,\mathrm{d}\tau\,. \tag{2}$$

Given a certain A, we could define a *family* of $\mathcal{W}$ which varies according to m. It can be proved that (1) could be *generically controllable* for any $m \geq 1$: on the other hand, it could be *practically uncontrollable* from an energetic point of view depending on the particular values of m. The energy necessary to steer the system from $x(0) = 0$ to a desired final value $x(t) = x_f$ is defined as

$$E(x_f, T) = \int_0^t \|u^o(\tau)\|^2\,\mathrm{d}\tau = x_f^T \mathcal{W}_t^{-1} x_f \tag{3}$$

where $u^o(\cdot)$ is the optimal input which steers the system from the initial state $x_0 = 0$ to $x(t) = x_f$ while minimizing the control energy. As a matter of fact, it is possible to define different controllability metrics which derives from the Controllability Gramian $\mathcal{W}_t$:

- **worst case energy:** simple calculation shows that
$$\max_{\|x_f\|=1} E(x_f, t) = \lambda_{min}(\mathcal{W}_t^{-1})$$
namely the the larger $\lambda_{min}(\mathcal{W}_{\mathcal{K},t})$ the more controllable the system;
- **average control energy:** it can be shown that
$$\mathrm{Tr}(\mathcal{W}_t^{-1})$$
measures the average control energy: in particular the smaller $\mathrm{Tr}(\mathcal{W}_t^{-1})$ the more controllable the system;
- **quality of complete controllability:** the determinant of $\mathcal{W}_t$ measures the *volume* of the hyperellipsoid of final states x_f reachable from the initial state $x_0 = 0$ and with control input energy $E(x_f, t) \leq 1$.

The analysis of the Controllability Gramian plays a fundamental role in assessing if a system is practically controllable or not. However, for large scale systems the computation of the Gramian (2) is computationally costly and typically suffers of *noise* and ill-conditionement. The computation of the metrics introduced before is thus a difficult task, subject to huge computational inaccuracy. Furthermore, if one seeks to solve an optimal control problem to steer a system to a desired final state x_f, solutions which depends on the Controllability Gramian would be really unreliable.

A. Output controllability

In many engineering applications we are interested in controlling an output of the system rather than its full state. A system

$$\Sigma_x\,:\,\begin{cases}\dot{x} = Lx + Bu\\ y = Cx\end{cases} \tag{4}$$

with $y \in \mathbb{R}^r$, is said to be output controllable if for any $y_f \in \mathbb{R}^r$ and $t > 0$ there exists an input function $u(\tau)$, $0 < \tau \leq t$, such that the output goes from $y(0) = 0$ to $y(t) = y_f$. This property can be checked through the rank of the output controllability matrix.

Theorem 1 *System* (4) *is output controllable if and only if the matrix $\mathcal{C}_{\mathcal{O}} \in \mathbb{R}^{r\times nm}$*

$$\mathcal{C}_{\mathcal{O}} = [CB \quad CLB \quad CL^2B \quad \ldots \quad CL^{n-1}B]$$

has full row rank.

In general, assuming output controllability is less restrictive than assuming (standard) controllability. In terms of energy, it seems plausible that controlling an output in place of each single state of the system would require less energy. As a matter of fact, the energy necessary to steer the output $y(0) = 0$ to a desired final value $y(t) = y_f$ can be expressed as a function of the Output Controllability Gramian

$$\mathcal{W}_o = C\,\mathcal{W}\,C^T\,. \tag{5}$$

where for the sake of simplicity in the notation we dropped the subscript $\mathcal{K}, t$.

Lemma 1 *Given a system in the form of* (4)*, the minimal energy necessary to steer the output $y(0) = 0$ to a desired final value $y(t) = y_f$ is given by*

$$E(y_f, t) = y_f^T\,\mathcal{W}_o^{-1}\,y_f \tag{6}$$

where $\mathcal{W}_o$ is (5)*.*

Proof: The minimal energy necessary to steer the output to the final value can be obtained by solving the optimal problem

$$\min_{\substack{u(\cdot)\,\text{s.t.}\\ y(t)=y_f}} \int_0^t u(\tau)^T u(\tau)\,\mathrm{d}\tau \tag{7}$$

We have

$$\mathcal{H} = u^T u + \lambda^T(Lx + Bu) \tag{8}$$
$$\mathcal{H}_x = \lambda^T L \tag{9}$$
$$\mathcal{H}_u = 2u^T + \lambda^T B \tag{10}$$

with constraint

$$\mathcal{X}(x_f)\,:\,\mathbb{R}^n \to \mathbb{R}^m, \qquad \mathcal{X}(x_f) = Cx \tag{11}$$

We seek a solution to the *two-boundary value problem*

$$\dot{x}^o = Lx^o + Bu^o \qquad x_0^o = x_0 \tag{12}$$
$$\dot{\lambda}^o = -\mathcal{H}_x^T \qquad \lambda_f^o = \mathcal{X}_x^T\nu \tag{13}$$

Namely we have

$$\dot{\lambda} = -\lambda^T L \qquad \Rightarrow \qquad \begin{aligned} \lambda(t) &= e^{-L^T t}\lambda_0 \\ \lambda_f &= e^{-L^T t_f}\lambda_0 = C^T \nu \end{aligned}$$

Thus we have $\lambda_0 = e^{L^T t_f} C^T \nu$ and as a consequence

$$\lambda(t) = e^{L^T (t_f - t)} C^T \nu \tag{14}$$

and as a consequence

$$\mathcal{H}_u = 0 \;\;\Rightarrow\;\; u = -\frac{1}{2}B^T\lambda = -\frac{1}{2}B^T e^{L^T(t_f-t)}C^T\nu \tag{15}$$

In order to find ν, we have

$$x(t) = e^{Lt}x_0 - \frac{1}{2}\left[\int_0^t e^{A(t-\tau)}BB^Te^{A^T(t-\tau)}\mathrm{d}\tau\right]C^T\nu$$

which for $t = t_f$ gives us r constraints

$$y_f = Ce^{Lt_f}x_0 - \frac{1}{2}C\left[\int_0^{t_f} e^{L(t_f-\tau)}BB^Te^{L^T(t_f-\tau)}\mathrm{d}\tau\right]C^T\nu$$

we obtain

$$\nu = 2\,\mathcal{W}_o\,\left[Ce^{At_f}x_0 - Cx_f\right] \tag{16}$$

Finally, we have

$$u^o(t) = -B^Te^{A^T(t_f-t)}C^T\mathcal{W}_o^{-1}\left[Ce^{At_f}x_0 - Cx_f\right] \tag{17}$$

For the sake of simplicity we set $x_0 = 0$ and by introducing (17) into (7) we obtain (6). $\triangleleft$

The analysis of the output Controllability Gramian defined in (5) allows to understand the input-output energy relationship for system (4).

At this stage, the Output Controllability Gramian $\mathcal{W}_o$ still requires the computation of the Controllability Gramian $\mathcal{W}$. In case of large scale networks, the problem introduce in previous sections are still present. In order to overcome them, in Section III we introduced a reduced model which allows to simplify the calculation of the Controllability Gramian and thus the Output Controllability Gramian and the minimal energy control input.

III. A REDUCED MODEL NETWORK

In this section we consider a generic network system in the form of (1). We suppose that a certain number m of nodes can be directly controlled. Thus system (4) can be expressed as

$$\begin{aligned} \dot{\begin{bmatrix} z \\ \delta \end{bmatrix}} &= \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}\begin{bmatrix} z \\ \delta \end{bmatrix} + \begin{bmatrix} I_m \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0_{1\times m} & H^T \end{bmatrix}\begin{bmatrix} z \\ \delta \end{bmatrix} \end{aligned} \tag{18}$$

with L Metzler, $z = [x_1, \ldots, x_m]^T$, $\delta = [x_{m+1}, \ldots, x_n]^T$ and H^T is a full vector, namely

$$H^T = \begin{bmatrix} h_1 & \ldots & h_{n-m} \end{bmatrix}, \qquad h_i \neq 0 \quad \forall\, i\,.$$

The partition of L, B and C follows the partition of the system state into $[z, \delta]^T$. In particular, z represents the nodes which can directly be controlled through the control input u while δ represents the uncontrolled nodes in the network. The output is then defined as a measurement of the uncontrolled nodes (for instance their average).

With the goal of controlling the output in mind, in order to simplify the computation of the Output Controllability Gramian and minimal energy control input, we aim to define a reduced model for (18). We consider the possibility to *aggregate* the dynamics of the uncontrolled nodes and to express the evolution of the output as a reduced dynamical system.

To this end, we define a matrix $P \in \mathbb{R}^{(m+1)\times n}$ as

$$P = \begin{bmatrix} I_m & 0 \\ 0 & H^T \end{bmatrix}. \tag{19}$$

Accordingly we introduce a new representation for (1)

$$\hat{x} = Px\,.$$

As a matter of fact, by considering (18) we can write

$$\begin{aligned} \dot{\hat{x}} &= \begin{bmatrix} L_{11} & L_{12} \\ H^TL_{21} & H^TL_{22} \end{bmatrix} x + \begin{bmatrix} I_m \\ 0 \end{bmatrix} u \\ &= \begin{bmatrix} L_{11} \\ H^TL_{21} \end{bmatrix} z + \begin{bmatrix} L_{12} \\ H^TL_{22} \end{bmatrix}\delta + \begin{bmatrix} I_m \\ 0 \end{bmatrix} u \end{aligned} \tag{20}$$

Furthermore, we have that

$$P^+ = \begin{bmatrix} I_m & 0 \\ 0 & \frac{1}{\sigma}H \end{bmatrix}, \qquad P^+P = \begin{bmatrix} I_m & 0 \\ 0 & \frac{1}{\sigma}HH^T \end{bmatrix}$$

where $\sigma = \|H\|^2$ and $P^+ \in \mathbb{R}^{n\times(m+1)}$ is such that $PP^+ = I_{m+1}$. As a consequence we can write

$$\begin{aligned} x &= P^+\hat{x} + (I_n - P^+P)x \\ &= \begin{bmatrix} I_m & 0 \\ 0 & \frac{1}{\sigma}H \end{bmatrix}\hat{x} + \begin{bmatrix} 0 & 0 \\ 0 & I_\eta - \frac{1}{\sigma}HH^T \end{bmatrix}\begin{bmatrix} z \\ \delta \end{bmatrix} \end{aligned}$$

where $\eta = n - m$ and which through simple computations leads to

$$\delta = \begin{bmatrix} 0 & \frac{1}{\sigma}H \end{bmatrix}\hat{x} + (I_\eta - \frac{1}{\sigma}HH^T)\delta$$

Thus (20) can be rewritten as

$$\begin{aligned} \dot{\hat{x}} &= \begin{bmatrix} L_{11} & \frac{1}{\sigma}L_{12}H \\ H^TL_{21} & \frac{1}{\sigma}H^TL_{22}H \end{bmatrix}\hat{x} + \begin{bmatrix} I_m \\ 0 \end{bmatrix} u \\ &\quad + \begin{bmatrix} L_{12}(I_\eta - \frac{1}{\sigma}HH^T) \\ H^TL_{22}(I_\eta - \frac{1}{\sigma}HH^T) \end{bmatrix}\delta \end{aligned} \tag{21}$$

System (21) is a *reduced model* for (1), namely $x \in \mathbb{R}^n$ and $\hat{x} \in \mathbb{R}^{m+1}$. In case y represents the average of the aggregated nodes, the third term in (21) can be seen as the variance of the measure of the states $[x_{m+1}, \ldots, x_n]$ with respect to y.

To simplify the computations in the following section, we set the control input u as

$$u = \begin{bmatrix} 0 & \frac{1}{\sigma}L_{12}H \end{bmatrix}\hat{x} + \upsilon \tag{22}$$

and rewrite (21) as

$$\begin{aligned} \dot{\hat{x}} &= \begin{bmatrix} L_{11} & 0 \\ H^TL_{21} & \frac{1}{\sigma}H^TL_{22}H \end{bmatrix}\hat{x} + \begin{bmatrix} I_m \\ 0 \end{bmatrix}\upsilon \\ &\quad + \begin{bmatrix} L_{12}(I_\eta - \frac{1}{\sigma}HH^T) \\ \frac{1}{\sigma}H^TL_{22}(I_\eta - \frac{1}{\sigma}HH^T) \end{bmatrix}\delta \end{aligned} \tag{23}$$

In compact form we rewrite (23) as

$$\begin{aligned} \dot{\hat{x}} &= \hat{L}\hat{x} + \hat{B}v + \hat{D}\delta \\ y &= \hat{C}\hat{x} \end{aligned} \tag{24}$$

where $\hat{C} = \begin{bmatrix} 0_{1\times m} & 1 \end{bmatrix}$. It is readily seen that the system is output controllable as long as $H^T L_{21} \neq 0$. Thus, by using Lemma 1, the following holds.

Lemma 2 *Given a system in the form of* (24)*, the minimal energy necessary to steer the output* $y(0) = 0$ *to a desired final value* $y(t) = y_f$ *is given by*

$$\hat{E}(y_f, t) = y_f^2\, \hat{\mathcal{W}}_o^{-1} \tag{25}$$

where $\hat{\mathcal{W}}_o$ *is Controllability Gramian of system* (24).

Lemma 2 holds for any system (24). In particular if $D = \mathbf{0}_{m+1\times\eta}$, the energy $E(y_f, t_f)$ for (18) and $\hat{E}(y_f, t)$ for (24) coincides. However the case of $D = \mathbf{0}_{m+1\times\eta}$, commonly referred to as *exact reduction*, holds only for particular structure of the system and particular choices of H (see [28] for more details on the subject).

We are thus interested in comparing the Output Controllability Gramian $\mathcal{W}_o$ for the original system (18) with the $\hat{\mathcal{W}}_o$ for (24). In the next Section we will show that $\hat{\mathcal{W}}_o$ displays some interesting properties and simulation results confirms that its properties are closely related to $\mathcal{W}_o$.

A. The Output Controllabilty Gramian of the reduced system

We recall that for a matrix $\hat{A} \in \mathbb{R}^{(m+1)\times(m+1)}$ with structure

$$\hat{A} = \begin{bmatrix} A & 0 \\ \mathbf{a}^T & \alpha \end{bmatrix} \tag{26}$$

where $A \in \mathbb{R}^{m\times m}$, $\mathbf{a} \in \mathbb{R}^{m\times 1}$ and $\alpha \in \mathbb{R}$, simple computations show that

$$\hat{A}^n = \left[\begin{array}{c:c} A^n & 0 \\ \hdashline \mathbf{a}^T \sum_{i=0}^{n-1} \alpha^{n-i-1} A^i & \alpha^n \end{array}\right]$$

As a consequence, by using the Taylor expansion

$$e^{\hat{A}t} = \sum_{i=0}^{\infty} \frac{\hat{A}^i t^i}{i!}$$

the following holds.

Lemma 3 *The exponential of* (26) *preserves a block triangular structure and in particular can be expressed as*

$$e^{\hat{A}t} = \begin{bmatrix} e^{At} & 0 \\ \Lambda & e^{\alpha t} \end{bmatrix} \tag{27}$$

where

$$\Lambda = \sum_{i=0}^{\infty} \mathbf{a}^T \frac{(\alpha I_m - A)^{-1}(\alpha^i I_m - A^i)}{i!} t^i .$$

Proof: We define $\mathcal{A}_1 = \mathbf{blkdiag}(A, \alpha)$ and

$$\mathcal{A}_2 = \begin{bmatrix} 0 & 0 \\ \mathbf{a}^T & 0 \end{bmatrix}$$

where $\mathcal{A}_2$ is nilpotent. We have that

$$e^{\hat{A}t} = \sum_{i=0}^{\infty} \frac{(\mathcal{A}_1 + \mathcal{A}_2)^i t^i}{i!}$$

Thus by considering $(\mathcal{A}_1 + \mathcal{A}_2)^i$ we have

$$\begin{aligned} (\mathcal{A}_1 + \mathcal{A}_2)^2 &= \mathcal{A}_1^2 + \begin{bmatrix} 0 & 0 \\ \alpha\mathbf{a}^T & \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mathbf{a}^T A & 0 \end{bmatrix} \\ (\mathcal{A}_1 + \mathcal{A}_2)^3 &= \mathcal{A}_1^3 + \begin{bmatrix} 0 & 0 \\ \alpha^2\mathbf{a}^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \alpha\mathbf{a}^T A & 0 \end{bmatrix} \\ &\qquad + \begin{bmatrix} 0 & 0 \\ \mathbf{a}^T A^2 & 0 \end{bmatrix} \\ \vdots \quad &= \quad \vdots \\ (\mathcal{A}_1 + \mathcal{A}_2)^i &= \mathcal{A}_1^i + \mathbf{a}^T(\alpha I_m - A)^{-1}(\alpha^i I_m - A^i) \end{aligned}$$

Then, by definition of $\mathcal{A}_1$ and $\mathcal{A}_2$, (27) follows straightforward. ◁

For the sake of compactness we rewrite $\hat{L}$ as

$$\hat{L} = \begin{bmatrix} \mathcal{L} & 0 \\ \boldsymbol{\ell}^T & \varphi \end{bmatrix}$$

and with a slight abuse of notation

$$e^{\hat{L}t} = \begin{bmatrix} e^{\mathcal{L}t} & 0 \\ \Lambda & e^{\varphi t} \end{bmatrix} .$$

By considering the definition given in (5), the Output Controllability Gramian of (24) is

$$\begin{aligned} \hat{\mathcal{W}}_o &= \hat{C}\,\hat{\mathcal{W}}\,\hat{C}^T \\ &= \hat{C}\left[\int_0^t e^{\hat{L}(t-\tau)}\hat{B}\hat{B}^T e^{\hat{L}^T(t-\tau)}\,\mathrm{d}\tau\right]\hat{C}^T \end{aligned} \tag{28}$$

In general, the (full) Controllability Gramian $\hat{\mathcal{W}}$ of (24) will have a structure of the form

$$\hat{\mathcal{W}} = \begin{bmatrix} \hat{\mathcal{W}}_{11} & \hat{\mathcal{W}}_{12} \\ \hat{\mathcal{W}}_{21} & \hat{\mathcal{W}}_{22} \end{bmatrix}$$

which, given definition of $\hat{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$, leads to the Output Controllability Gramian

$$\hat{\mathcal{W}}_o = \hat{\mathcal{W}}_{22}$$

As a matter of fact, considering the structure of $\hat{B}$ we have

$$\hat{B}\hat{B}^T = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}$$

and thus it follows

$$\hat{\mathcal{W}}_o = \boldsymbol{\ell}^T \tilde{\mathcal{L}} \int_0^t (e^{\varphi\tau} I_m - e^{\mathcal{L}\tau})(e^{\varphi\tau} I_m - e^{\mathcal{L}^T\tau})\mathrm{d}\tau\; \tilde{\mathcal{L}}^T \boldsymbol{\ell} \tag{29}$$

where we have defined $\tilde{\mathcal{L}} = (\varphi I_m - \mathcal{L})^{-1}$. The integral term reads as

$$\int_0^t e^{2\varphi\tau} I_m - e^{\varphi\tau I_m} e^{\mathcal{L}\tau} - e^{\varphi\tau I_m} e^{\mathcal{L}^T\tau} + e^{\mathcal{L}\tau} e^{\mathcal{L}^T\tau}\mathrm{d}\tau =$$

$$= \int_0^t e^{2\varphi\tau} I_m - e^{(\varphi I_m + \mathcal{L})\tau} - e^{(\varphi I_m + \mathcal{L}^T)\tau} + e^{\mathcal{L}\tau} e^{\mathcal{L}^T\tau}\mathrm{d}\tau$$

which, in case $\mathcal{L} = \mathcal{L}^T$ (the case of undirected graphs), becomes

$$\int_0^t e^{2\varphi\tau I_m} + e^{2\mathcal{L}\tau} - 2e^{(\varphi I_m + \mathcal{L})\tau} \mathrm{d}\tau \,.$$

As a consequence, the Output Controllability Gramian reads as

$$\hat{\mathcal{W}}_o = \boldsymbol{\ell}^T \tilde{\mathcal{L}} \left[\frac{e^{2\ell\tau I_m}}{2\varphi} + (2\mathcal{L})^{-1} e^{2\mathcal{L}\tau} \right. \\ \left. -2(\varphi I_m + \mathcal{L})^{-1} e^{(\varphi I_m + \mathcal{L})\tau} \right]_{\tau=0}^{t} \tilde{\mathcal{L}}^T \boldsymbol{\ell} \tag{30}$$

Since L is stable Metzler matrix, $\varphi < 0$ and $\mathcal{L}$ is Hurwitz: thus the first and third terms in the square brackets are vanishing. As a consequence, the second term is vanishing to. Then for $t \to \infty$ the Output Controllability Gramian converges to

$$\begin{aligned} \hat{\mathcal{W}}_o &= -\boldsymbol{\ell}^T \tilde{\mathcal{L}} \left[\frac{1}{2\phi} I_m + (2\mathcal{L}^{-1}) - 2(\varphi I_m + \mathcal{L})^{-1}) \right] \tilde{\mathcal{L}} \boldsymbol{\ell} \\ &= -\boldsymbol{\ell}^T \tilde{\mathcal{L}} \, \frac{(\varphi I_m - \mathcal{L})^2}{2\varphi \mathcal{L} (\varphi I_m + \mathcal{L})} \, \tilde{\mathcal{L}}^T \boldsymbol{\ell} \\ &= \boldsymbol{\ell}^T \, \Phi \, \boldsymbol{\ell} \end{aligned} \tag{31}$$

where Φ is defined as

$$\Phi = -\frac{1}{2\varphi} (\varphi \mathcal{L} + \mathcal{L})^{-1}$$

As far as $\boldsymbol{\ell}^T$ and Φ are concerned, we can observe that:

i) $\boldsymbol{\ell}^T \boldsymbol{\ell}$ express the relationship between the controlled nodes and the output. In networks, this boils down to saying that

$$\boldsymbol{\ell}^T \boldsymbol{\ell} \propto d_{z \to \delta} \tag{32}$$

where $d_{z \to \delta}$ is the out-degree from controlled nodes z to aggregated nodes δ;

ii) Φ can be seen as the network *inertia* of controlled nodes and the aggregated nodes: it can be bounded as

$$\|\Phi\| \leq \frac{1}{-2\varphi^2 \lambda_{min}(\mathcal{L}) - 2\varphi \lambda_{min}(\mathcal{L})^2} \tag{33}$$

where $\lambda_{min}(\mathcal{L})$ is the smallest eigenvalue of $\mathcal{L}$.

From **i)**, we can infer that controllability depends on the number of connections between controlled nodes and the aggregated nodes: the more connections, the less energy required to control the output of the aggregated nodes. From **ii)**, we can conclude that the smaller the time-constant of the aggregated nodes φ and the minimum eigenvalue of the controlled nodes $\lambda_{min}(\mathcal{L})$, the bigger Φ and thus the less the energy required to control the output of the aggregated nodes.

IV. Application to Average-Control of Large Scale Flow Networks

In this section we consider the application of the concept presented in this paper to the case of Large Scale flow networks. We consider networks of n nodes with a ratio of controlled nodes $\mathcal{K} = \frac{m}{n}$ that varies between $\mathcal{K} = \{\frac{1}{2}, \ldots, \frac{1}{40}\}$.

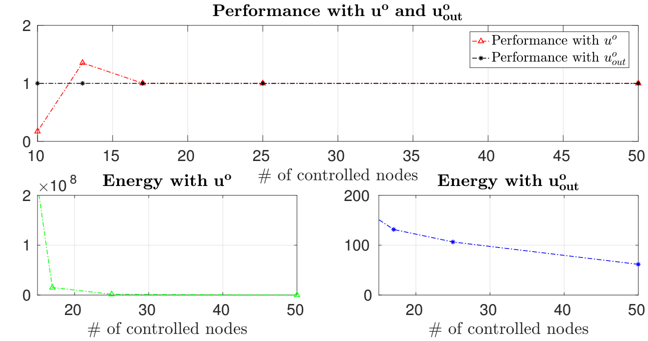


Fig. 1: Full Controllability and Output Controllability. As long as the ratio of controlled nodes is $\mathcal{K} \geq \frac{1}{5}$, full controllability attains the desired performance, however the energy required is much higher (10^5 order) than the case of output control. Once $\mathcal{K} < \frac{1}{5}$, the ill-conditioned Controllability Gramian makes the system diverge from the desired output and the energy grow even more.

The controlled nodes are selected for each network size and ratio as the one with lower *betweenness centrality*, namely the *boundaries* of the network. The network dynamics can be written as

$$\Sigma_x \, : \left\{ \begin{array}{ll} \dot{x} & = Lx + Bu \\ y & = Cx \end{array} \right. \tag{34}$$

where L is a Hurwitz Metlzer matrix: this choice is consistent with transportation networks where the dynamics of the nodes can be described by a conservation law (Laplacian matrix) with some additional dissipative terms on the boundaries (sinks of the network) [29]. Following the transportation networks application, the graph has a pseudo-normal distribution centered around 4, representing a Manhattan grid. The Laplacian matrix of the network is normalized and thus $|\lambda_i(L)| \leq 2$ [30].

By sorting the agents in such a way that the controlled nodes of the graph are separated from the others, we can rewrite Σ_x as (18) with

$$B = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \qquad\qquad C = \begin{bmatrix} 0_{1 \times m} & H^T \end{bmatrix} \tag{35}$$

In particular, we set $H^T = \frac{1}{\eta} \mathbb{1}^T$, namely the average of the uncontrolled nodes. This choice is motivated again by flow networks where typically we are allowed to control the boundaries of the graph with the goal to steer some measure of the interior nodes (in this case the average) to a desired value. In case of traffic networks for instance, the controlled nodes would be the *gates* to the networks and the output the average density of the interior nodes.

A. Full Controllability and Output Controllability

In order to justify the premise to this paper, we first show that in case of large scale networks Output Controllability is computationally much more convenient than Full Controllability. This advantage appears evident already for *small* networks.

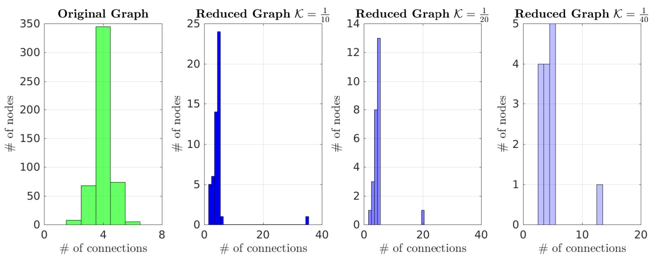


Fig. 2: Connection Distribution for the original system and of the reduced order system: the green histograms shows the original network, namely a network with pseudo-Normal Distribution for $n = 500$. The blue histograms represents the distribution of the network obtained after the model reduction is preformed for $\mathcal{K} = \{\frac{1}{10}, \frac{1}{20}, \frac{1}{40}\}$.

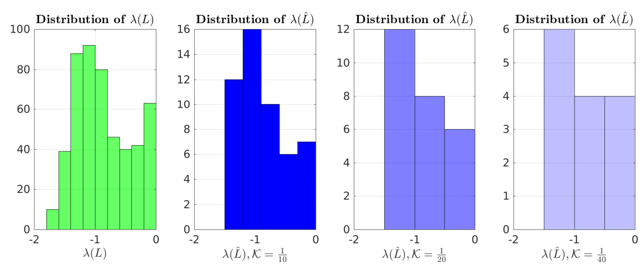


Fig. 3: Eigenvalue Distribution of L and $\hat{L}$: for a network of $n = 500$ nodes and $\mathcal{K} = \{\frac{1}{10}, \frac{1}{20}, \frac{1}{40}\}$. The spectrum of $\hat{L}$ covers the same spectrum of L for any $\mathcal{K}$.

We consider a network of $n = 100$ nodes and we compare the case of Full Controllability with the Output Controllability: we set a target value $y_f = 1$ and a $t_f = 10\,\text{sec}$ and seek for a solution to (7). The Full Controllability solution for system (34) is given by

$$u^o(t) = -B^T e^{L^T(t_f - t)} \mathcal{W}^{-1} \left[e^{L t_f} x_0 - \mathbb{1}_\eta y_f \right] \tag{36}$$

while the Output Controllability solution is given by

$$u^o_{out}(t) = -B^T e^{L^T(t_f - t)} C^T \mathcal{W}_o^{-1} \left[C e^{L t_f} x_0 - y_f \right] . \tag{37}$$

We define a performance measure as the ratio between the output obtained by setting the two control inputs just defined and the desired output y_f.

In **Figure 1**, it can be seen that as long as $\mathcal{K} \geq \frac{1}{5}$, the full controllability attains the same desired output of the output controllability however with a much higher energy. Once $\mathcal{K} < \frac{1}{5}$, the Controllability Gramian of the full system is ill-conditioned to the point of making the system unable to reach the desired value. The energy as well increases rapidly up to an unrealistic value. On the other hand, Output Controllability still reaches the desired value and the energy increases proportionally to the decrements of control nodes.

B. Boundary Control of Normally Distributed Graphs

We are now interested in considering the properties of the reduced system and to evaluate the performance obtained by designing control based on this approximation. In the case presented in this section, the reduced system (23) reads as

$$\hat{\Sigma}_x : \begin{cases} \dot{\hat{x}} &= \begin{bmatrix} L_{11} & L_{12}\mathbb{1} \\ \frac{1}{\eta}\mathbb{1}^T L_{21} & \frac{1}{\eta}\mathbb{1}^T L_{22}\mathbb{1} \end{bmatrix} \hat{x} + \begin{bmatrix} I_m \\ 0 \end{bmatrix} u \\ & \quad + \begin{bmatrix} L_{12}(I_\eta - \frac{1}{\eta}\mathbb{1}\mathbb{1}^T) \\ \frac{1}{\eta}\mathbb{1}^T L_{22}(I_\eta - \frac{1}{\eta}\mathbb{1}\mathbb{1}^T) \end{bmatrix} \delta \\ y &= \begin{bmatrix} 0_{1\times m} & 1 \end{bmatrix} \hat{x} \end{cases} \tag{38}$$

For a graph of $n = 500$ nodes, the connection distribution before and after the reduction is shown in **Figure 2**: as already mentioned the original network as a Manhattan *structure*, with degree centered around 4 connections. The distribution for (38) depends on the number of controlled nodes and here we display the case of $\mathcal{K} = \{\frac{1}{10}, \frac{1}{20}, \frac{1}{40}\}$. We can clearly see that the aggregated nodes become a unique node which possesses many connections towards the boundary nodes.

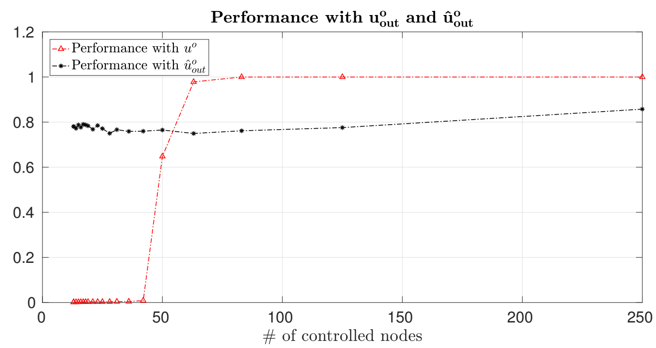


Fig. 4: Output of the system with two different control inputs u^o and $\hat{u}^o$, respectively calculated on the original system and on the reduced system for different size of the network. Control nodes are sitting on the boundary of the network.

By considering the reduced system $\hat{\Sigma}_x$, the computation of the Controllability Gramian $\hat{\mathcal{W}}$ becomes much easier and in turn the computation of the minimal energy control input $\hat{u}^o$. The solution to (7) for $\hat{\Sigma}_x$ is given by

$$\hat{u}^o_{out}(t) = -\hat{B}^T e^{\hat{L}^T(t_f - t)} \hat{C}^T \hat{\mathcal{W}}_o^{-1} \left[\hat{C} e^{\hat{L} t_f} \hat{x}_0 - y_f \right] . \tag{39}$$

We are interested in comparing the behavior of the system Σ_x in the case in which the control input u is designed according to (37) and in the case in which it is designed based on the reduce system, namely (39). We again set $y_f = 1$ and $t_f = 10\,\text{sec}$. For the sake of simplicity, we impose $x_0 = 0$ and consequently $\hat{x}_0 = 0$.

As shown in **Figure 3**, the specturm of $\hat{L}$ overlaps the spectrum of L fro any $\mathcal{K}$: clearly, due to the reduction of size from n to $m+1$, the spectrum of $\hat{L}$ has less eigenvalues but they cover the *same* interval. This is a crucial aspect since the input design highly depends on the spectrum of L and $\hat{L}$: with this respect it it worth pointing out that considering the normalized Laplacian matrix plays an important role in guaranteeing that the original spectrum and the reduced spectrum are covering the same *range* (a condition that otherwise would hold only for regular graphs). As a consequence, we can expect that (37) and (39) will have a *similar* behavior.

In order to compare the performance of the system with respect to the two inputs, we consider the ratio between the output of (34) when (37) is used and when (39) is used with the reference value $y_f = 1$. **Figure 4** shows the perfromance of these two cases for different values of control nodes m. As long as the $\mathcal{K} \geq \frac{1}{5}$ the control input (37) (red plot) outperforms (39) (black plot) of around 20%. However, we find out that around the ratio $\mathcal{K} = \frac{1}{6}$ a *phase-transition* appears in the controllability properties of a network.

In particular, when considering input (37) designed over the original system, we observe a critical change in the performance of the system. On the other hand, this phenomena does not appear when we design the input as (39), namely on the reduced system. The performance of the system with (37) degrade abruptly and greatly, when $\mathcal{K} \leq \frac{1}{6}$ while the behavior of the network with (39) is much more *regular*, namely the performance decreases proportionally to a decreasing number of control nodes.

The origin of this behavior seems to lie in the eigenvalue of the Controllability Gramian. The Controllability Gramian computed over (34) is *fragile* with respect to $\mathcal{K}$ and suddenly becomes inaccurate and highly ill-conditioned. On the other hand, the reduced system (38) is more robust and the control input (39) preserves acceptable performance for smaller $\mathcal{K}$. As reported in **Figure 5**, this behavior is scale-free with respect to the size of the network. For input (37), we can still observe the *phase-transition* appearing around $\mathcal{K} = \frac{1}{6}$ and an abrupt degradation of the performances for smaller number of control nodes. With input (39), the system display a much more *robust* behavior for smaller $\mathcal{K}$.

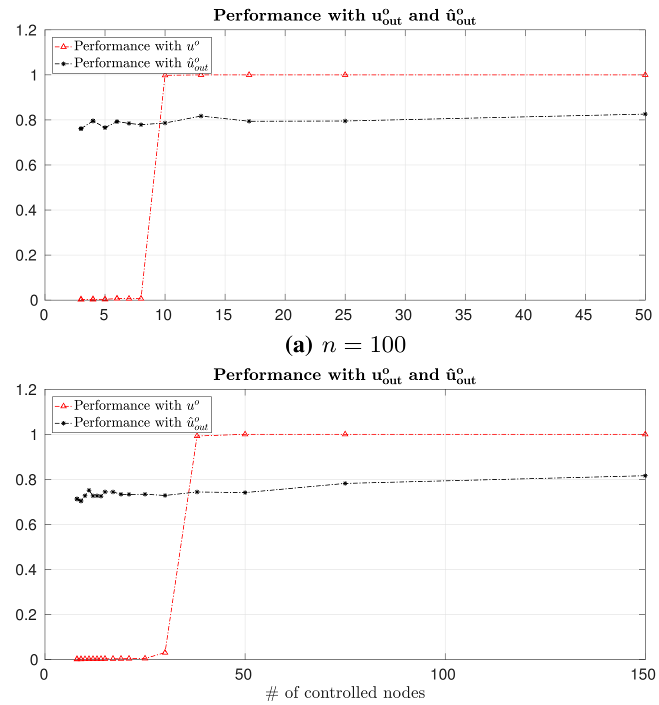
In a bigger picture, we can say that the design (39) is also scale-free also with respect to $\mathcal{K}$, namely the degradation of performance caused by a reducing number of controlled nodes is almost neglectable.

C. The out-degree of controlled nodes, the Inertia and the Control Energy

The the performance observed in previous simulations is consistent also with the evolutions of the Output Controllability Gramian $\mathcal{W}_o$ of the original system and $\hat{\mathcal{W}}_o$ of the reduce system. For a network of $n = 500$ nodes, **Figure 6** shows the ratio between between $\hat{\mathcal{W}}_o$ and $\mathcal{W}_o$. For any $\mathcal{K}$, $\hat{\mathcal{W}}_o$ underestimates $\mathcal{W}_o$ around 15%-20%.

Regarding the Output Controllability Gramian, at the end of Section III-A we showed that $\hat{\mathcal{W}}_o$ is closely related to the out-degree of the controlled nodes z towards the aggregated nodes δ (32) and to the network inertia (33). These two aspects, which can be easily inferred through the analysis of the reduced order system, are indeed confirmed by simulation results.

In **Figure 7**, we considered again a network of $n = 500$ nodes with a fixed number of controllers $m = 100$. We plot the performance of the system and the control energy with the two inputs (37) and (39) respectively, for increasing connectivity degrees $d_{z\to\delta}$. It can be seen that the performance



(a) $n = 100$

(b) $n = 300$

Fig. 5: Output of the system with two different control inputs u^o and $\hat{u}^o$, respectively calculated on the original system and on the reduced system for different size of the network, namely $n = 100$ and $n = 300$.

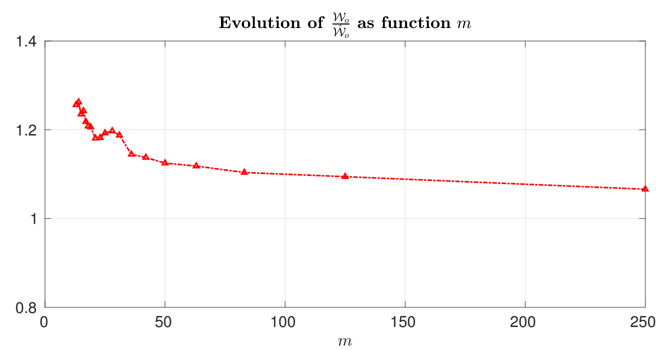


Fig. 6: Ratio between the Outptu Controllability Gramian of Σ_x and $\hat{\Sigma}_x$ calculated over 100 different graphs.

with input (39) improve for higher $d_{z\to\delta}$. Most importantly, the control energy decreases *proportionally* to an increasing $d_{z\to\delta}$. Similarly, in Figure 8, we plot performance and energy of the network with respect to $\|\Phi\|$ in (33). We can clearly see that for an increasing $\|\Phi\|$, the energy required to achieve the desired goal decreases greatly.

These simulations allows to say that the reduced system (38) is a good approximation of the original system (34) from an energetic perspective. As a matter of fact, the relationship between the control energy and the network structure (obtained in (32) and (33)) is shown to have a real impact.

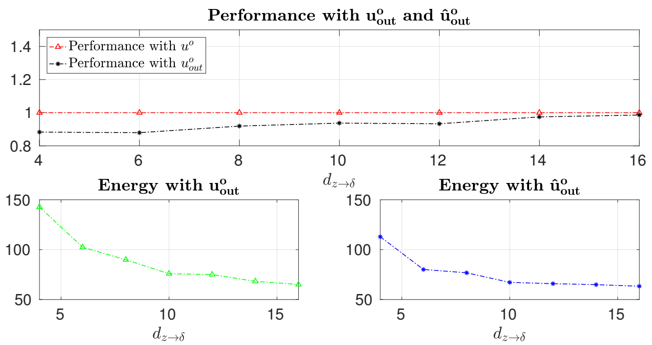


Fig. 7: Performance and Energy with respect to the degree of controlled nodes towards aggregated nodes: performances both with u^o_{out} and $\hat{u}^o_{out}$ increases. Most importantly, the control energy decreases with an increasing $d_{z\to\delta}$.

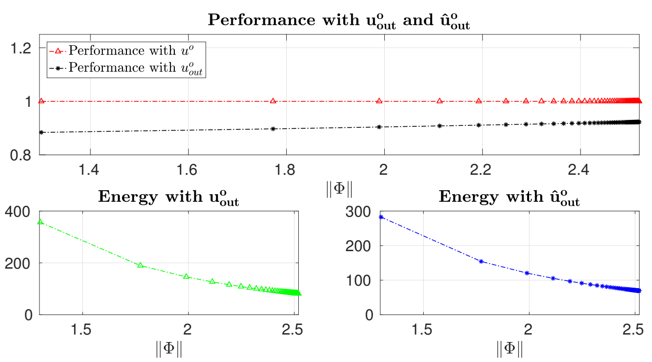


Fig. 8: Performance and Energy with respect to inertia of the network $\|\Phi\|$: performances both with u^o_{out} and $\hat{u}^o_{out}$ increases. Most importantly, the control energy decreases for an increasing $\|\Phi\|$.

V. Conclusion

In this paper we considered the problem of controlling a large scale network. Rather than controlling the full state, we consider the case in which we are interested in controlling an output which depends on the non-controlled nodes. Thanks to this choice, the network is practically controllable independently from the size.

In order to simplify the computation of the Controllability Gramian and minimum energy control inputs, we considered a model-reduced approximation of the system, base on the aggregation of the non-controlled nodes. We showed that this new systems is reliable both for estimating the energy necessary to control the output and also to design optimal control. The reduction of complexity due to the aggregation simplifies a lot the computation and ultimately shows to be more reliable than performing computation over the large original system.

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